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Oriented paths in n-chromatic digraphs

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Rajai Nasser

Abstract

In this thesis, we try to treat the problem of oriented paths in n-chromatic digraphs. We first treat the case of antidirected paths in 5-chromatic digraphs, where we explain El-Sahili's theorem and provide an elementary and shorter proof of it. We then treat the case of paths with two blocks in n-chromatic digraphs with n greater than 4, where we explain the two different approaches of Addario-Berry et al. and of El-Sahili. We indicate a mistake in Addario-Berry et al.'s proof and provide a correction for it.

Contents

Introduction

Gallai-Roy's celebrated theorem [8, 9] states that every n-chromatic digraph contains a directed path of length $n-1$. More generally, one can ask which connected digraphs are contained in every n-chromatic digraph. Such digraphs are called n-universal. Since there exist n-chromatic graphs with arbitrarily large girth [10], n-universal digraphs must be oriented trees. Burr [11] proved that every oriented tree of order n is $(n-1)^2$-universal (in particular every oriented path is $(n-1)^2$-universal) and he conjectured that every oriented tree of order n is $(2n-2)$-universal. This is a generalization of Sumner's conjecture which states that every oriented tree of order n is contained in every tournament of order $2n-2$. The first linear bound for tournaments was given by Häggkvist and Thomason [12]. The best bound so far, $3n-3$, was obtained by El Sahili [13].

Regarding oriented paths in general, there is no better result than the one given by Burr, that is every oriented path is $(n-1)^2$-universal. However in tournaments, Havet and Thomassé [14] proved that except for three particular cases, every tournament of order n contains every oriented path of order n.

El-Sahili showed [1] that except the regular 5-tournament T_5, any 5-chromatic oriented digraph in which each vertex has out-degree at least two, contains a copy of the anitidirected path p_4 of length 4. To show his result, El-Sahili used a theorem of Gallai [5]. In chapter II, we give a detailed explanation of El-Sahili's proof, we provide a new elementary shorter proof without using Gallai's theorem, and we conjecture a stronger statement.

El-Sahili conjectured [15] that every path of order $n \geq 4$ with two blocks is n-universal, and Bondy and El-Sahili [15] proved it if one of the two blocks has length one. The condition $n \geq 4$ is necessary because of odd circuits. El-Sahili and Kouider [16] introduced the notion of maximal spanning out-forests and used it to show a weak version of El-Sahili's conjecture which states that every path of order n with two blocks is $(n+1)$-universal.

L. Addario-Berry et al [2] used strongly connected digraphs and a theorem of Bondy [17] to show El-Sahili's conjecture. El-Sahili and Kouider [3] gave a new elementary proof without using strongly connected digraphs or Bondy's theorem. In chapter III we give a detailed explanation of both proofs, we show that there is a small error in Addario-Berry et al' proof and we provide a correction.

All the definitions and basic notations used in this master thesis will be explained in Chapter I.

Chapter 1

Definitions and basic notations

1.1 Graphs and multi-graphs

A *graph* is a pair $G = (V, E)$ of sets such that E is a subset of the power set $P(V)$ of V where every element of E contains exactly two elements of V. The elements of V are called the *vertices* of G and the elements of E are called *the edges* of G. The set of vertices of G is referred to as $V(G)$, and the set of edges is referred to as $E(G)$. An edge $\{x, y\}$ is noted by xy. *The order* $|G|$ of the graph G is the number of vertices in $V(G)$. A graph where we can find an edge between any two distinct vertices is called *complete*. A complete graph of order n is denoted K_n;

A *multi-graph* is a triplet $G = (V, E, \varphi)$ where V and E are two sets, and φ is a mapping from E into $P(V)$ such that for every e in E, $\varphi(e)$ contains one or two vertices of V. We say that V is the set of vertices of G and we write $V(G) = V$, similarly we say that E is the set of edges of G and we write $E(G) = E$. *The order of a multi-graph* is also the number of vertices in $V(G)$.

If e is an edge and $\varphi(e)$ contains only one vertex v we say that e is a *loop* on v. If e_1 and e_2 are two different edges on the same vertices i.e. $\varphi(e_1) = \varphi(e_2)$, we say that e_1 and e_2 are *parallel edges*. A multi-graph $G = (V, E, \varphi)$ without loops or parallel edges can be seen as a graph: we identify it with the graph $G' = (V, \varphi(E))$.

If G_1 and G_2 are two graphs such that $V(G_1) \subset V(G_2)$ and $E(G_1) \subset E(G_2)$ we say that G_1 is a *subgraph* of G_2. If in addition $E(G_1)$ contains all the edges xy of G_2 such that $x, y \in V(G_1)$, we say that G_1 is an *induced subgraph* of G_2, and we write $G_1 = G_2[V(G_1)]$. If G_1 is a subgraph of G_2 and $V(G_1) = V(G_2)$ we say that G_1 *spans* G_2.

A mapping $f : V(G_1) \longrightarrow V(G_2)$ is said to be a *morphism of graphs* if $\forall x, y \in V(G_1)$ we have $f(x)f(y) \in E(G_2)$ whenever $xy \in E(G_1)$. If f is injective, we say that G_2 contains a copy of G_1 which is $f(G_1) := (f(V(G_1)), \{f(x)f(y) \in E(G_2)/xy \in E(G_1)\})$, or for simplicity we may say that G_2 contains G_1. If f is bijective, we say that f is an *isomorphism of graphs* and that G_1 and G_2 are *isomorphic*.

If $G_1 = (V_1, E_1, \varphi_1)$ and $G_2 = (V_2, E_2, \varphi_2)$ are two multi-graphs, then we say that G_1 is a *sub-multi-graph* of G_2 if $V_1 \subset V_2$, $E_1 \subset E_2$ and φ_1 is the restriction of φ_2 on E_1.

1.2 Digraphs and oriented multi-graphs

A digraph is a pair $D = (V, E)$ of sets such that $E \subset V \times V$, and such that for every $(x, y) \in E$ we must have $(y, x) \notin E$, in particular if $(x, y) \in E$ then $x \neq y$. We call V the set of vertices of D and we write $V(D) = V$, similarly we call E is the set of arcs (or edges) of D and we write $E(D) = E$. If $e = (x, y) \in E$, we write $x \rightarrow y$; we say that x is the *tail* of e and we write $t(e) = x$ and we say that y is the *head* of e and we write $h(e) = y$. *The order of a digraph* is the number of vertices in $V(D)$.

If D_1 and D_2 are two digraphs such that $V(D_1) \subset V(D_2)$ and $E(D_1) \subset E(D_2)$ we say that D_1 is a *subdigraph* of D_2. If in addition $E(D_1)$ contains all the arcs (x, y) of D_2 such that $x, y \in V(D_1)$, we say that D_1 is an *induced subdigraph* of D_2, and we write $D_1 = D_2[V(D_1)]$. If D_1 is a subdigraph of D_2 and $V(D_1) = V(D_2)$ we say that D_1 *spans* D_2.

A mapping $f : V(D_1) \longrightarrow V(D_2)$ is said to be a *morphism of digraphs* if $\forall x, y \in V(D_1)$ we have $(f(x), f(y)) \in E(D_2)$ whenever $(x, y) \in E(D_1)$. If f is injective, we say that D_2 contains a copy of D_1 which is $f(D_1) := (f(V(D_1)), \{(f(x), f(y)) \in E(D_2)/(x, y) \in E(D_1)\})$, or for simplicity we may say that D_2 contains D_1. If f is bijective, we say that f is an *isomorphism of digraphs* and that D_1 and D_2 are *isomorphic*.

Let $D = (V, E)$ be a digraph. the *underlying graph* $G(D)$ of D is defined as $G(D) := (V, \psi(E))$, where $\psi : V \times V \longrightarrow P(V)$ is defined as $\psi((x, y)) = \{x, y\}$, $\forall x, y \in V$. A digraph whose underlying graph is complete is called a *tournament*.

An *Oriented multi-graph* is a triplet $D = (V, E, \varphi)$ where V and E are two sets, and φ is a mapping from E into $V \times V$. We say that V is the set of vertices of D and we write $V(D) = V$, similarly we say that E is the set of arcs (or edges) of D and we write $E(D) = E$. If $e \in E(D)$ and $\varphi(e) = (x, y)$ we write $x \rightarrow y$; we say that x is the *tail* of e and we write $t(e) = x$ and we say that y is the *head* of e and we write $h(e) = y$. *The order of an oriented multi-graph* is the number of vertices in $V(D)$.

The *underlying multi-graph* of an oriented multi-graph $D = (V, E, \varphi)$, is the multi-graph $G(D) = (V, E, \psi \circ \varphi)$ where $\psi : V \times V \longrightarrow P(V)$ is defined as $\psi((x, y)) = \{x, y\}$, $\forall x, y \in V$. If D is an oriented multi-graph whose underlying multi-graph is a graph (contains no loops and no parallel edges), D can be seen as a digraph: we identify D with $D' = (V, \varphi(E))$.

If $D_1 = (V_1, E_1, \varphi_1)$ and $D_2 = (V_2, E_2, \varphi_2)$ are two oriented multi-graphs, then D_1 is a *sub-oriented-multi-graph* of D_2 if $V_1 \subset V_2$, $E_1 \subset E_2$ and φ_1 is the restriction of φ_2 on E_1.

For simplicity, we will not be strict when dealing with graphs (resp. multi-graphs, oriented multi-graphs or digraphs), in the sense that if G is a graph (resp. multi-graph, oriented multi-graph or digraph) we may not differ strictly between G and $V(G)$ or between G and $E(G)$: If v is a vertex of G and e is an edge of G, we may write $v \in G$ and $e \in G$ rather than $v \in V(G)$ and $e \in E(G)$. Also if H is a subgraph (resp. sub-multi-graphs, sub-oriented-multi-graph or subdigraph) of G, and e is an edge (or arc) of G we denote by $H \cup e$ or $H + e$ the subgraph (resp. sub-multi-graphs, sub-oriented-multi-graph or subdigraph) of G obtained from H by adding the edge e, and if $e \in H$ we denote by $H - e$ the subgraph (resp. sub-multi-graphs, sub-oriented-multi-graph or subdigraph) of G obtained from H by deleting the edge e.

1.3 Degree and neighborhood of a vertex

Let G be a graph, if $e = xy$ is an edge of G and we say that the vertices x and y are *adjacent* and we say that e is *incident* to x and y. *The neighborhood* $N(v)$ of a vertex v is defined as the set of vertices adjacent to it, and its *degree* $d(v)$ is the number of vertices in $N(v)$ which is equal to the number of edges incident to v.

Let G be a graph. The *maximum degree* of G is defined as $\Delta(G) := max\{d(v)/v \in V(G)\}$ and the *minimum degree* of G is defined as $\delta(G) := min\{d(v)/v \in V(G)\}$.

Let $D = (V, E)$ be a digraph. *The neighborhood* $N(v)$ of a vertex v is its neighborhood in the underlying graph. *The degree* $d(v)$ of a vertex v is its degree in the underlying graph. the *out-neighborhood* $N^+(v)$ of a vertex v is defined as $N^+(v) := \{w \in V(D)/v \to w\}$. Similarly the *in-neighborhood* $N^-(v)$ is defined as $N^-(v) := \{w \in V(D)/w \to v\}$. The *out-degree* $d^+(v)$ of a vertex v is the number of arcs whose tail is v, and the *in-degree* $d^-(v)$ of v is the number of arcs whose head is v. We define also $\Delta^+(D) := max\{d^+(v)/v \in V(D)\}$, $\delta^+(D) := min\{d^+(v)/v \in V(D)\}$, $\Delta^-(D) := max\{d^-(v)/v \in V(D)\}$ and $\delta^-(D) := min\{d^-(v)/v \in V(D)\}$.

1.4 Paths and cycles

Let G be a graph (or multi-graph), a *path* P from x to y in G is a finite sequence $P = x_1x_2...x_n$ (we can have $n = 1$) of distinct vertices such that $x_1 = x$, $x_n = y$ and $x_ix_{i+1} \in E(G)$ for $1 \leq i \leq n-1$, x_1 and x_n are the *end vertices* of P. A *sub-path* of a path P is a path which is a subset of P. A *cycle* C in G is a finite sequence $C = x_1x_2...x_n$ (we can have $n = 1$) of distinct vertices such that $x_nx_1 \in E(G)$ and $x_ix_{i+1} \in E(G)$ for $1 \leq i \leq n-1$. If $C = x_1...x_n$ is a cycle, then if $e = x_ix_j \in E(G)$ such that $i - j \neq 1 \bmod n$ and $j - i \neq 1 \bmod n$, then e is called a *chord* of C, if such chord does not exist we say that C is *chordless*. A graph (or multi-graph) is said to be *acyclic* if it does not contain any cycle; note that an acyclic multi-graph is necessarily a graph. *The length* $l(P)$ (resp. $l(C)$) of a path P (resp. a cycle C) is the number of edges in it. A *hamiltonian* path P (resp. cycle C) is a path (resp. cycle) which spans G, i.e. $V(P) = V(G)$ (resp. $V(C) = V(G)$).

The distance $d(x, y)$ between two vertices x and y, is the minimal length of a path from x to y if such path exists. If there is no path between x and y, we set $d(x, y) := \infty$. The map $d : V(G) \times V(G) \longrightarrow \mathbb{R} \cup \{\infty\}$ verifies the axioms of generalized metric, and so $(V(G), d)$ is a generalized metric space. *The diameter* $d(G)$ of a graph (or multi-graph) G, is the maximal distance between two vertices of G.

The girth $g(G)$ of a graph (or multi-graph) G is the minimal length of a cycle in it if such one exists, and if G is acyclic we set $g(G) := \infty$. If $g(G) = 1$ then G contains a loop and if $g(G) = 2$ then G contains parallel edges, so if $g(G) \geq 3$, G contains no loops and no parallel edges and so G is necessarily a graph. Note that all the above definitions are also defined for oriented multi-graphs by applying them on the underlying multi-graphs.

Let D be a digraph. A *directed path* P in D is a finite sequence of different vertices $P = x_1x_2...x_n$ (we can have n=1) such that $x_i \to x_{i+1}$ for $1 \leq i \leq n-1$. A *block of a path* P in D is a maximal directed sub-path of P. A path having l blocks of consecutive lengths $k_1, k_2, ..., k_l$

is denoted by $P^+(k_1, k_2, ..., k_l)$ (or $P(k_1, k_2, ..., k_l)$) if $x_1 \to x_2$ and $P^-(k_1, k_2, ..., k_l)$ if $x_1 \leftarrow x_2$. An *antidirected* path is a path whose blocks are all of length 1. A *circuit* C in D is a finite sequence of different vertices $C = x_1x_2...x_n$ (we can have n=1) such that $x_i \to x_{i+1}$ for $1 \leq i \leq n-1$ and $x_n \to x_1$.

1.5 Connectivity

Let G be a graph (or multi-graph), G is *connected* if $d(G) < \infty$, i.e. there exist a path between any two vertices. G is *disconnected* if it is not connected. G is *k-connected* if it remains connected after the removal of any $k' < k$ vertices. The *connectivity* $\kappa(G)$ is the maximal integer k such that G is k-connected ($\kappa(G) = 0$ if and only if G is disconnected). G is *k-edge-connected* if it remains connected after the removal of any $k' < k$ edges. The *edge-connectivity* $\lambda(G)$ is the maximal integer k such that G is k-edge-connected ($\lambda(G) = 0$ if and only if G is disconnected).

Let G be a graph (or multi-graph), a maximal connected subgraph of G is called a *connected component* of G. Suppose that G is connected, a vertex v whose removal disconnect G is a *cut-vertex* of G and an edge e whose removal disconnect G is a *bridge* of G. A maximal connected subgraph of G without cut-vertices is called a *block* of G.

Let D be a digraph (resp. oriented multi-graph), all the above notations are defined for D by applying them on its underlying graph (resp. underlying multi-graph). D is called *strongly connected* if by choosing any two vertices x and y in D we can find a directed path from x to y and a directed path from y to x.

1.6 Trees and forests

An acyclic graph is called a *forest*. A connected acyclic graph is called a *tree*, so a forest is the union of trees (each graph is the union of its connected components). The vertices of degree 1 in a tree are called the *leaves* of the tree. A tree containing only one vertex is called a *trivial tree*, then a non-trivial tree contain at least two leaves (consider for example the ends of a longest path).

The following assertions are equivalent for a graph T (the proof is straightforward for the first four, use a simple induction for the last two):

1. T is a tree.
2. Any two vertices of T are linked by a unique path.
3. T is minimally connected, i.e. T is connected and $T - e$ is disconnected for all edges $e \in E(T)$.
4. T is maximally acyclic, i.e T is acyclic and $T + xy$ contains a cycle for any two non-adjacent vertices x and y of G.
5. T is connected and $|E(T)| = |V(T)| - 1$.
6. T is acyclic and $|E(T)| = |V(T)| - 1$.

An *oriented tree* is a digraph whose underlying graph is a tree, similarly an *oriented forest* is a digraph whose underlying graph is a forest. An *out-leaf* of a tree is a leaf whose out-degree is zero, similarly an *in-leaf* of a tree is a leaf whose in-degree is zero. An *out-branching* (resp. *in-branching*) is an oriented tree in which a unique vertex which we call *the root* has its in-degree (resp. out-degree) 0, and the other vertices has in-degree (resp. out-degree) 1. An *out-forest* (resp. *in-forest*) is an oriented forest whose connected components are out-branchings (resp. in-branchings). Let F be an out-forest, the level $l_F(v)$ of a vertex $v \in F$ is the order of a longest directed path ending at v.

1.7 Coloring

A *k-coloring* of a graph (or multi-graph) G is a mapping $c : G \longrightarrow \{1, 2, ..., k\}$ (we can use any set of k elements instead of $\{1, 2, ..., k\}$). If v is a vertex of G we say that $c(v)$ is the color of v, and if v is adjacent to a vertex of color i, we say that v is adjacent to the color i. A *good k-coloring* of a graph G is a coloring c such that any adjacent vertices does not have the same color.

If G admits a good k-coloring, we say that G is *k-colorable*. A subset L of $V(G)$ is said to be *stable* if there is no adjacent vertices in it, i.e. the set of edges in the subgraph $G[L]$ of G induced by L is empty. G is said to be *independent* if $V(G)$ is stable. Note that G is k-colorable if and only if we can partition $V(G)$ into k stable subsets.

The *chromatic number* $\chi(G)$ of G is the least integer k such that G is k-colorable. If $\chi(G) = k$ and $\chi(G - v) < k \ \forall v \in V(G)$ we say that G is *k-critical*. All the above notations can be defined for digraphs (resp. oriented multi-graphs) by applying them on their underlying graphs (resp. underlying multi-graphs).

1.8 Contraction and minors

Let G be a graph, and let H be a subset of $V(G)$ (or a subgraph of G), then the graph obtained from G by *contracting* H is G/H defined by $V(G/H) := (V(G) \setminus H) \cup \{v_H\}$ where v_H is a new vertex and $E(G/H) := \{xy/xy \in E(G),\ x, y \in V(G) \setminus H\} \cup \{vv_H/v \in V(G) \setminus H,\ \exists v' \in H,\ vv' \in E(G)\}$.

Let $D = (V, E)$ be a digraph, and let H be a subdigraph of D, We say that D is *contractable* by H if for all vertices v in $V(D) \setminus V(H)$, we cannot find two arcs $v \rightarrow x$ and $y \rightarrow v$ such that $x \in H$ and $y \in H$, i.e. all arcs in D between v and H are in the same direction. In this case, the digraph obtained from D by *contracting* H is D/H defined by $V(D/H) := (V(D) \setminus H) \cup \{v_H\}$ where v_H is a new vertex and $E(D/H) := \{(x, y)/(x, y) \in E(D),\ x, y \in V(D) \setminus H\} \cup \{(v_H, v)/v \in V(G) \setminus H,\ \exists v' \in H,\ (v', v) \in E(G)\} \cup \{(v, v_H)/v \in V(G) \setminus H,\ \exists v' \in H,\ (v, v') \in E(G)\}$.

Let $D = (V, E, \varphi)$ be an oriented multi-graph, and let H be a sub-oriented-multi-graph of D, then the oriented multi-graph obtained from D by *contracting* H is $D/H = (V', E', \varphi')$

where $V' = (V \setminus V(H)) \cup \{v_H\}$, $E' = E \setminus E(H)$ and $\varphi' : E' \longrightarrow P(V')$ is defined by $\varphi'(e) = (f_H(t_D(e)), f_H(h_D(e)))\ \forall e \in E'$ where $f_H(x) = x$ if $x \notin H$ and $f_H(x) = v_H$ if $x \in H$.

Note that the notation D/H have different meaning when interpreting D as a digraph or oriented multi-graph. The notation takes its meaning relatively to the context.

If G is a graph (resp. digraph or oriented multi-graph) and if G' is a graph (resp. digraph or oriented multi-graph), we say that G' is a *minor* of G if there exist a finite sequence $G_1, G_2, ..., G_n$ of graphs (resp. digraph or oriented multi-graph) such that $G_1 = G$, $G_n = G'$ and $\forall i \in \{1, 2, ..., n-1\}$ G_{i+1} is a subgraph (resp. subdigraph or sub-oriented-multi-graph) of G_i or obtained from G_i by contracting some subgraph (resp. subdigraph or sub-oriented-multi-graph) of it.

Chapter 2

Antidirected paths in digraphs

2.1 Introduction

The antidirected path p_4 is a digraph defined, up to isomorphism as follows:

$$V(p_4) = \{x, y, z, v, w\}, E(p_4) = \{(y, x), (y, z), (v, z), (v, w)\}$$

Let T_5 be the 5-tournament satisfying $d^+(u) = d^-(u) = 2\ \forall u \in T_5$. Grunbaum [5] proved that T_5 is the only 5-tournament which doesn't contain a copy of p_4. El-Sahili [1] showed that except T_5, any 5-chromatic oriented digraph in which each vertex has out-degree at least two, contains a copy of p_4. He showed by an example that the condition that each vertex has out-degree at least two is necessary.

To show his result, El-Sahili used a theorem of Gallai [6], which states that if G is k-critical, then each block of the subgraph of G induced by the vertices of degree $k-1$, is either complete or chordless odd cycle.

In this chapter we will give a detailed explanation of the argument used by El-Sahili to show his theorem. We will then provide a new elementary shorter proof which does not require the use of Gallai's theorem. We conclude this chapter by stating a new conjecture generalizing this theorem.

2.2 First Step of the proof

Theorem 2.1 [1]: *Let D be a 5-chromatic connected digraph distinct from T_5 in which each vertex has out-degree at least two. Then D contains a copy of p_4.*

To prove this theorem, we need several lemmas:

Lemma 2.2 [6]: *Except for T_5, any 5-tournament contains a copy of p_4.*

Proof: Suppose to the contrary that there exists a 5-tournament T other then T_5 which does not contain any p_4, then there exist at least one vertex v_1 in T such that $d^+(v) \geq 3$, let $\{v_2, v_3, v_4\} \subset N^+(v)$. we can assume without loss of generality that we have $v_2 \to v_3 \to v_4 \to v_2$ or $v_2 \to v_3 \to v_4 \leftarrow v_2$.

In the first case, if $\exists i \in \{2, 3, 4\}$ such that $v_i \to v_5$, we may assume without loss of generality

that $i = 2$, then $v_5v_2v_3v_1v_4$ is a p_4, so we conclude that $v_5 \to v_i, \forall i \in \{2, 3, 4\}$, but in this case $v_3v_5v_2v_1v_4$ would be a p_4.

In the latter case, we have $v_5 \to v_2$ because otherwise $v_5v_2v_3v_1v_4$ is a p_4. If $\exists i \in \{3, 4\}$ such that $v_5 \to v_i$, we may assume without loss of generality that $i = 3$ and so $v_3v_5v_2v_1v_4$ is be a p_4, so we conclude that $\forall i \in \{3, 4\}, v_i \to v_5$, but in this case $v_5v_3v_4v_1v_2$ is a p_4. □

Corollary 2.3: *If D is a digraph verifying the conditions of Theorem 2.1 and if D contains a K_5, then D contains a copy of p_4.*

Proof: Let p_3 be the subpath of p_4 formed by the first three edges, and let p_2 be the subpath of p_4 formed by the first two edges. Since $G(D)$ contains K_5, then D contains a 5-tournament. If this 5-tournament is not T_5 then by *Lemma 2.2* we conclude that D contains a copy of p_4.

Then we may assume that D contains T_5, and since D is not exactly T_5, then we will have an edge xy in $G(D)$ such that x is outside T_5 and y belongs to T_5. If $y \to x$ then this edge along with a path p_3 in T_5 starting at y (we can always find a copy of p_3 in T_5 starting at any point of it), form a path p_4. Otherwise, since $d^+(x) \geq 2$, then there exist a vertex z distinct from y such that $x \to z$. If $z \notin T_5$, then the path zxy along with a copy of p_2 in T_5 starting at y, form a copy of p_4. If $z \in T_5$, then the path zxy along with a copy of p_2 in T_5 starting at y and not intersecting z (For any two vertices of T_5, we can always find a copy of p_2 starting at one vertex and not intersecting the other), form a copy of p_4. □

Theorem 2.4 [7]: *If G is a connected graph which is not complete nor an odd cycle, then $\chi(G) \leq \Delta(G)$.*

Corollary 2.5 [4]: *If D is a digraph which does not contain any tournament of order $2n+1$ $(n \geq 2)$, and in which any vertex has in-degree at most n, then $\chi(D) \leq 2n$.*

Proof: Suppose to the contrary that $\chi(D) \geq 2n+1$ and let D' be a $2n+1$-critical subdigraph of D. If there exists a vertex v in D' such that $d^+_{D'}(v) < n$ then $d(v) < 2n$, and since D' is $2n+1$-critical then $\chi(D' - v) = 2n$ and we can easily check that $\chi(D') = 2n$ (extend a good $2n$-coloring of $D' - v$ by giving v a color not adjacent to it; we can find such color since v is adjacent to at most $2n - 1$ vertices) which contradicts the fact that $\chi(D') = 2n + 1$.

We conclude that for every vertex in D' we have $d^+_{D'}(v) \geq n$ and $d^-_{D'}(v) \leq n$, and since $\sum_{v \in D'} d^+_{D'}(v) = \sum_{v \in D'} d^-_{D'}(v) = |E(D')|$ we conclude that for every vertex v in D' we have $d^+_{D'}(v) = d^-_{D'}(v) = n$ and so $d_{D'}(v) = 2n$ which implies that $\Delta(D') = 2n$. Obviously $G(D')$ is not an odd cycle since $\Delta(D') = 2n \geq 2 \times 2 = 4$, and it is not complete since otherwise D would contain a $2n + 1$-tournament, so by Brooks theorem (Theorem 2.4) we conclude that $\chi(D') \leq \Delta(D') = 2n$ which contradicts the fact that $\chi(D') = 2n + 1$. □

Lemma 2.6: *If D is a connected digraph in which each vertex has in-degree at most one. Then D contains at most one cycle which is a circuit.*

Proof: Let C be a cycle which is a subdigraph of D, if C is not a circuit then $\exists v \in C$ such that $d^-_C(v) \neq 1$ or $d^+_C(v) \neq 1$. Since $d_C(v) = 2$ and $d^-_C(v) \leq 1$ then we have $d^-_C(v) = 0$ and $d^+_C(v) = 2$, but we have $\displaystyle\sum_{v \in C} d^+_C(v) = \sum_{v \in C} d^-_C(v) = n$ then $\exists w \in C$ such that $d^-_C(w) = 2$ which

is a contradiction. We conclude that every cycle in D is necessarily a circuit.

Suppose that there exist two different circuits C_1 and C_2 subdigraphs of D. Suppose that $C_1 \cap C_2 \neq \phi$, since $C_1 \neq C_2$ then we can say without loss of generality that $C_2 \nsubseteq C_1$ and thus $\exists v \in C_2 \setminus C_1$ such that $\exists w \in C_1$ with $v \to w$, but w has another in-neighbor in C_1 which is a contradiction. We conclude that $C_1 \cap C_2 = \phi$, but D is connected then there exists a path between a vertex of C_1 and a vertex of C_2, let $P = x_1x_2...x_n$ be a minimal such path ($x_1 \in C_1$ and $x_n \in C_2$). P is minimal, so x_1 is the only vertex of P in C_1 and x_2 is the only vertex of P in C_2. Since x_1 has an in-neighbor in C_1 and $d^-(x_1) \leq 1$ we have $x_1 \to x_2$, let i be the maximum integer such that $x_i \to x_{i+1}$. If $i < n-1$ then $x_i \to x_{i+1}$ and $x_{i+2} \to x_{i+1}$ which contradicts the fact that $d^-(x_{i+1}) \leq 1$, so $i = n-1$ and $x_{n-1} \to x_n$ but x_n has another in-neighbor in C_2 which contradicts the fact that $d^-(x_n) \leq 1$. □

Note that the above corollary and lemma holds also when we substitute "in-degree" by "out-degree".

In the sequel, D denote an oriented digraph verifying the conditions of *theorem 2.1*. We suppose to the contrary that D does not contain any copy of p_4. by the above corollary we may assume that D does not contain any 5-tournament. Let D' be a 5-critical subdigraph of D and let D^o be the subdigraph of D' induced by the vertices of out-degree at least three in D', i.e. $D^o = \{x \in D'/d^+_{D'} \geq 3\}$.

Lemma 2.7: *D' contains at least one vertex whose out-degree in D' is at least three, i.e. D^o is not empty.*

Proof: Otherwise we would have $d^+_{D'}(v) \leq 2$ for every vertex v in D'. D, and hence D', contains no 5-tournament, so by *corollary 2.5* we conclude that $\chi(D') \leq 4$, which is a contradiction. □

Lemma 2.8: *Every vertex in D has at most one in-neighbor in D^o.*

Proof: Suppose to the contrary that there exists a vertex v having two in-neighbors $x, y \in D^o$ and let $\{v, x_1, x_2\} \subset N^+(x)$. If $y \in \{x_1, x_2\}$, we may suppose without loss of generality that $y = x_1$, then $\exists y_1 \in N^+(y) \setminus \{v, x, x_1, x_2\}$ since $d^+_{D'}(y) \geq 3$, thus y_1yvxx_2 is a p_4, a contradiction. So $y \notin \{x_1, x_2\}$ and more generally we can say that x and y are not adjacent. $d^+(y) \geq 3$ so $\exists y_1 \in N^+(y) \setminus \{x_1, x, v\}$ thus x_1xvyy_1 is a p_4, a contradiction. □

Corollary 2.9: $\forall v \in D^o, d^-_{D^o}(v) \leq 1$.

Proof: Clear. □

Lemma 2.10: *Let v be a vertex of D such that $d^+(v) \geq 3$ and $\{x, y, z\} \subset N^+(v)$. If $x \to y$ then $x \to z$, $yz \notin E(G(D))$ and $N^-(y) = N^-(z) = \{v, x\}$.*

Proof: If $x \nrightarrow z$ then $\exists w \in N^+(x) \setminus \{v, y, z\}$ since $d^+(v) \geq 2$, so $wxyvz$ is a p_4 which is a contradiction. So we must have $x \to z$.
If we suppose that $yz \in E(G(D))$, we may assume without loss of generality that $y \to z$. We have $d^+(y) \geq 2$ so $\exists w \in N^+(y) \setminus \{v, x, y, z\}$ and then $wyzvx$ is a p_4, a contradiction. So we have $yz \notin E(G(D))$.

Suppose that $N^-(y) \neq \{v, x\}$, so $\exists w \in N^-(y) \setminus \{v, x, y, z\}$, and since $d^+(w) \geq 2$ then $\exists w' \neq y$ such that $w \to w'$. If $w' = v$ then $vwyxz$ is a p_4, a contradiction. So $w' \neq v$, let $u \in \{x, z\} \setminus \{w'\}$, then $w'wyvu$ is a p_4, a contradiction. So $N^-(y) = \{v, x\}$ (We prove similarly that $N^-(z) = \{v, x\}$). □

Lemma 2.11: *If v and v' are two vertices such that there exist two adjacent vertices x and y in $N^+(v) \cap N^-(v')$, then $N^+(v) = \{x, y\}$.*

Proof: We may assume without loss of generality that $x \to y$. If $N^+(v) \neq \{x, y\}$ then $\exists w \in N^+(v) \setminus \{x, y\}$, by *lemma 2.10* w cannot be v', and so $wvyxv'$ is a p_4 which is a contradiction. □

Lemma 2.12: *D^o is an independent set of D.*

Proof: Suppose to the contrary that D^o is not an independent set, so there exist a connected component L of D^o which contains at least two vertices. If L is a circuit, then every vertex of L has one in-neighbor in L and has at least two out-neighbors outside D^o since its out-degree in D' is at least 3. If L is not a cycle, let v be the last vertex in a maximal directed path in L, we can easily verify that $d^-_{D^o}(v) = 1$ and $d^+_{D^o}(v) = 0$ so v has at least three out-neighbors outside D^o since $d^+_{D'}(v) \geq 3$.

So in all cases, we can always find a vertex v in L having at least two out-neighbors outside D^o and such that $d^-_L(v) = 1$, let v' be the in-neighbor of v in L and let v_1, v_2 and v_3 three out-neighbors of v in D' such that v_1 and v_2 are outside D^o (i.e. $d^+_{D'}(v_1) \leq 2$ and $d^+_{D'}(v_2) \leq 2$). D' is 5-critical, so any vertex in D' has at least 4 neighbors, we conclude that $d^-_{D'}(v_1) \geq 2$ and $d^-_{D'}(v_2) \geq 2$. If v_1 and v_2 are not adjacent, v_1 has one in-neighbor in $D' \setminus \{v, v_1, v_2\}$, otherwise we may assume without loss of generality that $v_1 \to v_2$, but since $d^-_{D'}(v_1) \geq 2$ we conclude again that v_1 has one in-neighbor in $D' \setminus \{v, v_1, v_2\}$. So in all cases we can say without loss of generality that there exist a vertex u in $D' \setminus \{v, v_1, v_2\}$ such that $u \to v_1$.

Suppose that $u = v_3$, by *Lemma 2.10* we have $v_3 \to v_2$ and by *Lemma 2.8* we have $u \notin D^o$ so $d^-_{D'}(u) \geq 2$ which implies that $\exists w \in D' \setminus \{v, v_1, v_2, v_3\}$ such that $w \to u$. Since $d^+(w) \geq 2$, there exists a vertex $w' \neq u$ such that $w \to w'$, suppose that $w' \neq v$ then let $w'' \in \{v_1, v_2\} \setminus \{w'\}$ so $w'wuvw''$ is a p_4, a contradiction. So we have $w \to v$. $d^+_{D'}(v') \geq 3$ then let $\{v'_1, v'_2, v\} \subset N^+_{D'}(v')$, so by *lemma 2.8* $w \neq v'$ and $\{v'_1, v'_2\} \cap \{v_1, v_2, v_3\} = \phi$. Let $w'' \in \{v'_1, v'_2\} \setminus \{w\}$, then $w''v'vwu$ is a p_4, a contradiction.

We conclude that $u \notin \{v, v_1, v_2, v_3\}$ and by *lemma 2.8* u cannot be v'. $d^+(u) \geq 2$ so there exist a vertex u' different from v_1 such that $u \to u'$. If $u' \neq v$, let $w \in \{v_2, v_3\} \setminus \{u'\}$, so $u'uv_1vw$ is a p_4, a contradiction. So $u' = v$, and by *lemma 2.11* we cannot have $v' \to u$, and since $d^+(v') \geq 3$, $\exists w \in N^+(v') \setminus \{u, v_1, v\}$ so $wv'vuv_1$ is a p_4, a contradiction. □

Lemma 2.13: *Let $v \in D^o$, then v has exactly three out-neighbors v_1, v_2 and v_3 in D' such that $v_1 \to v_2$ and $v_1 \to v_3$.*

Proof: Suppose that any two out-neighbors of v in D' are not adjacent, and let v_1, v_2 and v_3 be three out-neighbors of v in D'. $\forall i \in \{1, 2, 3\}, \exists u_i \in D' \setminus \{v, v_1, v_2, v_3\}$ such that $u_i \to v_i$. By *Lemma 2.8* we cannot have $u_1 = u_2 = u_3$ because otherwise we would have $u \in D^o$, so we may assume without loss of generality that $u_1 \neq u_2$. Suppose $u_1 \nrightarrow v$, we have $d^+(u_1) \geq 2$

so $\exists w \in D \setminus \{v_1, u_1, v\}$, let $w' \in \{v_2, v_3\} \setminus \{w\}$ so wu_1v_1vw' is a p_4 which is a contradiction. So we conclude that $u_1 \to v$ and similarly $u_2 \to v$ but we will have a copy of p_4 which is $v_1u_1vu_2v_2$, a contradiction. So we conclude that at least two in-neighbors of v, say v_1 and v_2, are adjacent, we may suppose that $v_1 \to v_2$, so by *lemma 2.10* we have also $v_1 \to v_3$.

Suppose that v has four out-neighbors v_1,v_2,v_3 and v_4 in D'. By *lemma 2.10* we have $v_1 \to v_3$ and $v_1 \to v_4$, thus $v_1 \in D^o$ and $\{v, v_1\} \subset N^-(v_2)$ which gives a contradiction with *lemma 2.8*. $\square$

2.3 Second Step (El-Sahili's proof)

In the sequel, we will need the use of the following theorem proved by Gallai:

Theorem 2.14 [5]: *Let G be a k-critical graph, then each block of the subgraph of G induced by the vertices of degree $k-1$ is either complete or a chordless odd cycle.*

Let D_m be the subdigraph of D' induced by the vertices of degree 4.

Lemma 2.15: *Any vertex v of D' has at least two in-neighbors in D'.*

Proof: If $v \in D' \setminus D^o$, then $d_{D'}(v) \geq 4$ because D' is 5-critical, and since $d^+_{D'}(v) \leq 2$ then $d^-_{D'}(v) \geq 2$. So we may assume that $v \in D^o$, let $N^+_{D'}(v) = \{v_1, v_2, v_3\}$ where $v_1 \to v_2$, $v_1 \to v_3$, $v_2v_3 \notin E(G(D))$ and $N^-(v_2) = N^-(v_3) = \{v, v_1\}$ (By *lemmas 2.10 and 2.13*). $\forall w \in N^-_{D'}(v_1) \setminus \{v\}$, we have $w \to v$ because otherwise we can find $u \in N^+(w) \setminus \{v, v_1\}$ and we can find $w' \in \{v_2, v_3\} \setminus \{u\}$, so $w'vv_1wu$ is p_4, which is a contradiction. If we suppose that $d^-_{D'}(v) = 1$ we will have $d^-_{D'}(v_1) = 2$ and so $d_{D'}(v) = d_{D'}(v_1) = d_{D'}(v_2) = d_{D'}(v_3) = 4$. Thus v_1, v_2, v_3 and v_3 are in the same block of D_m, this block cannot be an odd cycle, so by *theorem 2.14* it's complete which contradicts the fact that $v_2v_3 \notin E(G(D))$. $\square$

We now associate to each vertex v in D^o the set $S(v) = \{t(v), t'(v), v_0, ..., v_{g(v)}, v_{g(v)+1}\}$ $(0 \leq g(v) \leq 5)$, defined as follows: $N^+_{D'}(v) = \{v_0, t(v), t'(v)\}$ where $v_0 \to t(v)$ and $v_0 \to t'(v)$, $v_1 = v$; Set $T(v) = \{t(v), t'(v)\}$. If $d^-_{D'}(v_0) \geq 3$, put $g(v) = 0$; if not, let v_2 be the unique vertex of D' distinct from v_1 such that $v_2 \to v_0$. We have $v_2 \to v_1$. Again, if $d^-_{D'}(v_1) \geq 3$, put $g(v) = 1$; otherwise, let v_3 be the unique vertex of D' distinct from v_2 such that $v_3 \to v_1$; such a vertex exists by the above lemma. We have $v_2 \to v_1$, since otherwise we would have a path p_4 in D.

We may continue this process until meeting a vertex of in-degree at least three in D'; call this vertex $v_{g(v)}$, where $g(v)$ is the number of iterations required. Such a vertex exists and $g(v) \leq 5$. In fact suppose that $v_1, ..., v_5$ are defined as above and $d^-_{D'}(v_i) = 2, \forall i \in \{1, 2, 3, 4\}$. By *lemma 2.11* we have $d^+_{D'}(v_i) = 2, \forall i \in \{2, 3, 4, 5\}$. If $d^-_{D'}(v_5) = 2$, the vertices v_2, v_3, v_4 and v_5 will be in the same block of D_m. The block of D_m containing $\{v_2, v_3, v_4, v_5\}$ cannot be an odd cycle nor complete since $v_2v_5 \notin E(G(D))$ which contradicts *theorem 2.14*. Set

$O(v) = \{z \in D'/z \neq v_{g(v)+1} \text{ and } z \to v_{g(v)}\}$; we have $z \to v_{g(v)+1}$ for every z in $O(v)$.

Lemma 2.16: *If u and v are two distinct vertices of D^o then $S(u) \cap S(v) = \phi$.*

Proof: Let $S(v) = \{t(v), t'(v), v_0, ..., v_{g(v)}, v_{g(v)+1}\}$ and $S(u) = \{t(u), t'(u), u_0, ..., u_{g(u)}, u_{g(u)+1}\}$ and suppose that $\exists w \in S(u) \cap S(v)$. If $w = v$ then since u and v are not adjacent we should have $w \notin \{u, u_0, t(u), t'(u)\}$, otherwise we would have $u = v$ or $u \to v$. So $w = v \in S(u) \setminus \{u, u_0, t(u), t'(u)\}$ so $w = v = u_i$ with $i \geq 2$ which is a contradiction since $d^+_{D'}(v) = 3$ and $d^+_{D'}(u_i) = 2$. So we conclude that $w \neq v$ and similarly $w \neq u$. If $w \in \{v_0, t(v), t'(v)\}$ (i.e. $v \to w$), then $w \notin T(u)$ since otherwise we would have $v \in N^-(w) = \{u_0, u_1\} \subset S(u)$ (which is a contradiction), and similarly if $w = u_0$ we will have $w \notin T(v)$ and so $u_0 = w = v_0$ which implies that $N^+_{D'}(w) = N^+_{D'}(u_0) = T(u) = N^+_{D'}(v_0) = T(v)$ which is also a contradiction. We conclude that if $w \in \{v_0, t(v), t'(v)\}$ then $g(u) \geq 2$ and $w = u_i$ with $i \geq 2$; and more precisely we have $i = g(u)$ or $i = g(u) + 1$, because otherwise we would have $v \in N^-_{D'}(w) = N^-_{D'}(u_i) = \{u_{i+1}, u_{i+2}\} \subset S(u)$ which is a contradiction. Since $v \in N^-_{D'}(w)$ and $N^-_{D'}(u_{g(u)+1}) = N^-_{D'}(g(u)) \setminus \{u_{g(u)+1}\} = O(u)$, $d^+_{D'}(u_{g(u)+1}) = 2$ and $d^+_{D'}(v) = 3$ we conclude that $v \neq u_{g(u)+1}$ and then $v \in O(u)$. Since $v \in O(u)$ then $v \to u_{g(u)+1}$ and $v \to u_{g(u)}$, but $u_{g(u)+1} \to u_{g(u)}$, we can easily conclude that $v_0 = u_{g(u)+1}$ and then $N^+_{D'}(v_0) = \{t(v), t'(v)\} = N^+_{D'}(u_{g(u)+1}) = \{u_{g(u)}, u_{g(u)-1}\}$ which is a contradiction since $u_{g(u)} \to u_{g(u)-1}$ and $t(v)t'(v) \notin E(G(D))$. So $w \notin \{v_0, v_1, t(v), t'(v)\}$ and similarly $w \notin \{u_0, u_1, t(u), t'(u)\}$, so $\exists i \geq j \geq 2$ such that $w = u_i = v_j$; we will prove by induction on $0 \leq l \leq j-2$ that $u_{i-l} = v_{j-l}$: it is true for l=0, suppose that it is true for $l < j-2$ so $u_{i-l} = v_{j-l}$ which implies that $N^+_{D'}(u_{i-l}) = \{u_{i-l-1}, u_{i-l-2}\} = N^+_{D'}(v_{j-l}) = \{v_{j-l-1}, v_{j-l-2}\}$, but $u_{i-l-1} \to u_{i-l-2}$ and $v_{j-l-1} \to v_{j-l-2}$ so $u_{i-(l+1)} = v_{j-(l+1)}$. Set $l = j-2$, we conclude that $v_2 = u_{i-j+2}$ but this implies that $v \in N^+_{D'}(v_2) = N^+_{D'}(u_{i-j+2}) \subset S(u)$ which is a contradiction. $\square$

Lemma 2.17: *Let $L = \{v_{g(v)}/v \in D^o\}$. We have $\forall v \in L, d^-_{D'}(v) = 3$ and $\forall v \in D' \setminus L, d^-_{D'}(v) = 2$.*

Proof: Let $s = |L| = |D^o|$ and $p = |D' \setminus L| = |D' \setminus D^o|$, we have:

$$3s + 2p \leq \sum_{v \in D^o} d^+_{D'}(v) + \sum_{v \in D' \setminus D^o} d^+_{D'}(v) = \sum_{v \in D'} d^+_{D'}(v)$$

$$|E(D')| = \sum_{v \in D'} d^+_{D'}(v) = \sum_{v \in D'} d^-_{D'}(v)$$

$$\sum_{v \in D'} d^-_{D'}(v) = \sum_{v \in L} d^-_{D'}(v) + \sum_{v \in D' \setminus L} d^-_{D'}(v) \leq 3s + 2p$$

So we conclude that all the inequalities are in fact equalities, which holds only if we have $\forall v \in L\ d^-_{D'}(v) = 3$, $\forall v \in D' \setminus L\ d^-_{D'}(v) = 2$, $\forall v \in D^o\ d^+_{D'}(v) = 3$ and $\forall v \in D' \setminus D^o\ d^+_{D'}(v) = 2$. $\square$

Corollary 2.18: *For all $v \in D^o$, $O(v)$ contains exactly two vertices.*

Proof: Clear, by the definition of $O(v)$, and by *lemma 2.17*. $\square$

Proof of Theorem 2.1: Define the sets:

$$S = \bigcup_{v \in D^o} S(v), O = \bigcup_{v \in D^o} O(v), T = \bigcup_{v \in D^o} T(v)$$

We have $|O| \leq |T|$. Suppose that $O = T$, then $D' = D'[S]$ because otherwise we can find a vertex w outside S which is adjacent to a vertex v of S (D' is connected since it is 5-critical) and so $w \in N_{D'}(v)$ which means that $w \in S$ or $w \in O$ (see the definitions of $S(v)$, $O(v)$ and $T(v)$), then since $O = T \subset S$ then in all cases we have $w \in S$ which is a contradiction. Let v be a vertex of D^o, then put $c(t(v)) = c(t'(v)) = 1$, $c(v_0) = 2$ and $c(v_1) = 3$. If $g(v) = 0$ we are done, otherwise the colors 1,2 and 3 suffice to color the vertices of $S(v) \setminus \{v_{g(v)}, v_{g(v)+1}\}$, let $i \in \{2,3\} \setminus \{c(v_{g(v)-1})\}$ then put $c(v_{g(v)}) = 4$ and $c(v_{g(v)+1}) = i$. We can easily check that c is a good 4-coloring of the 5-chromatic digraph D' which is a contradiction.

So $O \neq T$ which means that $O \nsubseteq T$ or $T \nsubseteq O$. Since $|O| \leq |T|$ then $T \nsubseteq O$, and so we can find a vertex in T which is not in O. So we can find a vertex $v \in D^o$ such that $t(v) \notin O(v)$ or $t'(v) \notin O(v)$. We can assume without loss of generality that $t(v) \notin O(v)$ which implies that $N^+_{D'}(t(v)) \cap S = \phi$. Let $N^+_{D'}(t(v)) = \{u, u'\}$, $\{u, u'\} \cap (D^o \cup L) = \phi$ so $d^+_{D'}(u) = d^-_{D'}(u) = d^+_{D'}(u') = d^-_{D'}(u') = 2$. If u and u' are not adjacent, we can find $w \in D' \setminus \{t(v), u, u'\}$ such that $w \to u$, and if they are adjacent we can assume without loss of generality that $u \to u'$ and so again we can find $w \in D' \setminus \{t(v), u, u'\}$ such that $w \to u$. So without losing generality we can say that in all cases we can find $w \in D' \setminus \{t(v), u, u'\}$ such that $w \to u$. $w \nrightarrow t(v)$ since otherwise $w \in N^-_{D'}(t(v)) = \{v_0, v_1\}$ and $u \in N^+_{D'}(w) \subset \{v_0, t(v), t'(v)\}$, but $t(v) \to u$ and $v_0 \to t(v)$ so $u \neq v_0$ and $u \neq t(v)$, then $u = t'(v)$ which is contradiction since $t(v)t'(v) \notin E(G(D))$, and so $w \nrightarrow t(v)$. Since $d^+_{D'}(w) \geq 2$, $\exists w' \in D' \setminus \{u, w, t(v)\}$ such that $w \to w'$. If $w' \neq u'$, $w'wut(v)u'$ would be a p_4 which is a contradiction. We conclude that $N^+_{D'}(w) = \{u, u'\}$, and we have also $w \notin L$ because otherwise we would have $u \in S$. Then u,u',$t(v)$ and w are of degree 4, and so they are in the same block of D_m which cannot be neither an odd cycle nor complete which contradicts *theorem 2.14*. □

2.4 Our new shorter proof

We provide a new shorter proof of El-Sahili's theorem, which is elementary in the sense that it does not use Gallai's theorem. We will use all the theorems, lemmas and corollaries of the first step.

New proof of theorem 2.1: For all v in D^o we define v', $t(v)$ and $t'(v)$, such that $N^+_{D'}(v) = \{v', t(v), t'(v)\}$, $v' \to t(v)$ and $v' \to t'(v)$. Let $S(v) = \{v\} \cup N^+_{D'}(v)$, $H(v) = \{v, v'\}$, $O(v) = N^-_{D'}(v') \setminus \{v\}$ and $P(v) = N^-_{D'}(v) \setminus O(v)$. Note that $O(v)$ is not empty since $d^-_{D'}(v') \geq 2$ while $P(v)$ can be empty. $\forall w \in O(v), w \to v$ because otherwise $w'wv'vw''$ would be a p_4 where $w' \in N^+_{D'}(w) \setminus \{v', v\}$ and $w'' \in \{t(v), t'(v)\} \setminus \{w'\}$. By *lemma 2.12*, every vertex in $O(v)$ has only two out-neighbors i.e. v and v', in particular $O(v)$ is stable.

If $P(v)$ is not empty then $\forall w \in P(v), \exists w' \in O(v)$ such that $w \to w'$, since otherwise $d^+(w') \geq 2$ implies that $\exists w' \in D' \setminus (O(v) \cup \{v, v', w\})$ such that $w \to w'$ which means that $w'wvuv'$ is a p_4 where $u \in O(v)$. By *lemma 2.12*, every vertex in $P(v)$ has only two out-neighbors i.e. v and one vertex in $O(v)$, in particular $P(v)$ is stable.

Let $D^o = \{v_1, v_2, ..., v_l\}$, we define D_i, $S_i(v)$, $O_i(v)$ and $P_i(v)$ for $0 \leq i \leq l$ and $v \in D^o$ as follows: $D_0 = D'$, $S_0(v) = S(v)$, $O_0(v) = O(v)$ and $P_0(v) = P(v)$. D_{i+1}, $S_{i+1}(v)$, $O_{i+1}(v)$ and $P_{i+1}(v)$ are obtained from D_i, $S_i(v)$, $O_i(v)$ and $P_i(v)$ by removing $S_i(v_{i+1})$ and then

contracting $O_i(v_{i+1})$ and $P_i(v_{i+1})$ if any of them is not empty.

We can easily check that all the vertices of D_l has at most two out-neighbors. Suppose that D_l contains a 5-tournament T, then T contains at least one contracted vertex w (Otherwise T would be a subdigraph of D'). $w = v_{O_{l-1}(v)}$ or $v_{P_{l-1}(v)}$ for some $v \in D^o$, and in both cases w has at most one out-neighbor in D_l, and this means that:

$$10 = |E(T)| = \sum_{u \in T} d_T^+(u) = d_T^+(w) + \sum_{u \in T \setminus \{w\}} d_T^+(u) \leq 1 + 4 \times 2 = 9$$

Which gives a contradiction. So D_l does not contain any 5-tournament and by *corollary 2.5* we conclude that $\chi(D_l) \leq 4$.

Let i be the least integer such that $\chi(D_i) \leq 4$, then $i > 0$ because $D_0 = D'$ is 5-critical. Let $v = v_i$ and let c be a good 4-coloring of D_i. Color the vertices in $O_{i-1}(v)$ by $c(v_{O_{i-1}(v)})$ and color those in $P_{i-1}(v)$ by $c(v_{P_{i-1}(v)})$.

If $t(v)$ and $t'(v)$ are adjacent (in D_{i-1}) to at most three colors, we color them by a remainder color, then similarly color v and then v' (They are each adjacent to at most three colors) and we get $\chi(D_{i-1}) \leq 4$, a contradiction.

We conclude that $t(v)$ and $t'(v)$ are adjacent to the four colors 1,2,3 and 4. We may assume without loss of generality that $t(v)$ is adjacent to 1 and 2 and that $t'(v)$ is adjacent to 3 and 4. If $O_{i-1}(v) = \phi$, color $t(v)$ by 3, $t'(v)$ by 1, v by 2 and v' by 4, and we get good 4-coloring. So $O_{i-1}(v) \neq \phi$, we may assume without loss of generality that $c(v_{O(v)}) = 1$. Color $t(v)$ by 3 and color $t'(v)$ by 1, v by 2 and v' by 4, and we get good 4-coloration and $\chi(D_{i-1}) \leq 4$, a contradiction. □

2.5 Conclusion

In this chapter we have presented El-Sahili's theorem [1] stating that we can always find a copy of the anti-directed path p_4, in any 5-chromatic digraph where every vertex has at least two out-neighbors and which is not exactly T_5. We have presented El-Sahili's proof and we have provided a new shorter proof.

Is the condition that every vertex has at least two out-neighbors really necessary? El-Sahili gave a positive answer in his paper through the following example: Construct a digraph by adding to T_5 an arc (x, y) where $x \notin T_5$ and $y \in T_5$, then we can easily check that this digraph does not contain a copy of p_4.

The example given above contains T_5 and this shows that the condition that every vertex has at least two out-neighbors is necessary for digraphs containing T_5. What if it does not contain T_5? El-Sahili concluded his paper [1] by asking the following question: *Can we find a 5-chromatic digraph which contains neither a 5-tournament nor p_4?*

We conclude this chapter by stating the following conjecture of us:

Conjecture 2.19: *Let D be a $2n+1$-chromatic graph where $n \geq 2$. If D does not contain any $2n+1$-tournament, and if every vertex of D has at least n out-neighbors. Then D contains the antidirected path p_{2n} of length $2n$ starting with a backward arc.*

Chapter 3

Paths with two blocks in n-chromatic digraphs

3.1 Introduction

An important problem in graph theory is to find which oriented paths can be found in n-chromatic digraphs. Gallai-Roy's celebrated theorem [8, 9] states that every n-chromatic digraphs contains a directed path of length $n-1$. The question is that can we find an oriented path of length $n-1$ with more than one block? or more generally, how big should be the chromatic number of a digraph to guarantee the existence of an oriented path of length $n-1$?

Burr [11] proved that every $(n-1)^2$-chromatic digraph contains any tree of order n, in particular every $(n-1)^2$-chromatic digraph contains any oriented path of length n. In this chapter we are interested in paths with two blocks. El-Sahili [15] introduces the function $f(n)$ which is defined to be the minimal integer $f(n)$ such that every $f(n)$-chromatic digraph contains any path with two block $P(k,l)$ with $k+l=n-1$, and he conjectured that $f(n)=n$ for $n \geq 4$. El-Sahili proved [15] that $f(n) \leq \frac{3}{4}n^2$. El-Sahili and Bondy [15] proved that the conjecture holds when one of the two blocks have length 1.

El-Sahili and Kouider [16] introduced the notion of maximal spanning out-forest and used it to prove that $f(n) \leq n+1$. Addario-Berry et al [2] used strongly connected digraphs and maximal spanning out-forests to prove El-Sahili's conjecture ($f(n)=n$ for $n \geq 4$). Later El-Sahili and Kouider [3] provided a new elementary proof of El-Sahili's conjecture without using strongly connected digraphs. In this chapter we provide a detailed explanation of both methods. We show that the first method contains a small error and we provide a correction.

3.2 Maximal spanning out-forest

The level $l_F(v)$ of a vertex v in an out-forest F is defined as in the case of out-branching; the order of a longest directed path ending at v. We denote by $T_v(F)$ the out-branching of F rooted at v and by P_v the directed path in F of order $l_F(v)$ which ends at v. For all $u \in P_v$, P_uv denotes the uv-directed path in F.

Let D be a digraph, a spanning subdigraph F of D is said to be a *maximal spanning out-*

forest if F is a out-forest such that $\forall x, y \in V(D)$, if $x \to y$ with $l_F(x) \geq l_F(y)$ then there exists a directed path from y to x in F, i.e. $y \in P_x$. The set L_i of vertices having the same level i is a stable (by definition).

Let F be an out-forest which is a spanning subdigraph of a digraph D. If F is not a maximal out-forest, then there exist an arc $x \to y$ such that $l_F(x) \geq l_F(y)$ and there is no directed path from y to x in F, the out-forest F' obtained from F by deleting the arc whose head is y (If such one exists) and adding the arc $x \to y$ is called an *elementary improvement* of F.

We can easily see that the level of each vertex in F' is at least its level in F, and there exists a vertex (y) whose level strictly increases. Since the level of a vertex cannot increase infinitely (The maximum level that can be reached is —V(D)—), we can see that after a finite number of elementary improvements we get to a maximal spanning out-forest which is call a *maximal closure* of F. Thus starting with a spanning out-forest that contains no arcs we can prove the existence of a maximal spanning out-forest of D. We have also another way to get the existence of a maximal spanning out-forest; choose an out-forest F which maximizes the sum of the levels of all vertices.

The notion of maximal spanning out-forests introduced by El-Sahili and Kouider [16] is useful in the context of universal digraphs. As shown by El-Sahili and Kouider [16], it gives an easy proof of Gallai-Roy's theorem. Indeed, consider a maximal spanning out-forest of an n-chromatic digraph D. Since every level is a stable set, there are at least n levels. Hence D contains a directed path of length at least $n-1$. Final forests are also useful for finding paths with two blocks, as illustrated by the following proof due to El-Sahili and Kouider [16].

Lemma 3.1 [16]: *Let F be a maximal spanning out-forest of a digraph D. If $v \to w$ is an arc from F_i to F_j. Then*

1. *If $k \leq i < j - l$, then D contains a $P(k, l)$.*
2. *If $k < j \leq i - l$, then D contains a $P(k, l)$.*

Proof: 1. Let P_l be the directed path of F which starts at F_{j-l} and ends at w and P_{k-1} be the directed path in F starting at $F_{i-(k-1)}$ and ending at v. Then $P_{k-1} \cup vw \cup P_l$ is a $P(k, l)$.

2. Let P_{l-1} be the directed path in F which starts at F_{i-l+1} and ends at v. Let P_k be the directed path in F starting at F_{j-k} and ending at w. Then $P_k \cup P_{l-1} \cup vw$ is a $P(k, l)$. □

Corollary 3.2 [16]: *Every digraph with chromatic number at least $k + l + 2$ contains a $P(k, l)$.*

Proof: 1. Let F be a maximal spanning out-forest of D. Color the levels $F_1, ..., F_k$ of F with colors $1, ..., k$. Then color the level F_i, where $i > k$, with color $j \in \{k + 1, ..., k + l + 1\}$ such that $j \equiv i \mod l + 1$. Since this is a $k + l + 1$-coloring, it's not a good, and so there exists an arc which satisfies the hypothesis of *Lemma 2.3*. □

3.3 Paths with two blocks in strongly connected digraphs

Theorem 3.3 [17]: *Every strongly connected digraph D has a circuit of length at least $\chi(D)$.*

Let k be a positive integer and D be a digraph. A directed circuit C of D is k-good if $|C| \geq k$ and $\chi(D[V(C)]) \leq k$. Note that Theorem 3.3 states that every strongly connected digraph D has a $\chi(D)$-good circuit.

Note that the last part of the proof in [2] of the following lemma contains an error. We will show the proof in [2] and explain why it is false, and then we will provide a correction.

Lemma 3.4: *Let D be a strongly connected oriented multi-graph and $k \in \{3, ..., \chi(D)\}$. Then D has a k-good circuit.*

Proof: By Bondy's theorem, there exists a circuit with length at least $\chi(D)$, so the lemma is true for $k = \chi(D)$. If $k = 3$ then if C is the shortest circuit of D, then it's chordless and therefore $\chi(C) = 2$ or 3. Suppose that $3 \leq k < \chi(D)$ and consider a shortest circuit C with length at least k. We claim that $\chi(D[V(C)]) \leq k$.Suppose to the contrary that $\chi(D[V(C)]) \geq k+1$, and let D' be a maximal sub-oriented-graph of $D[V(C)]$ such that D' is a strongly connected digraph in which C is a subdigraph. If any two vertices of D' are adjacent in D, they are still adjacent in D', and so $\chi(D') = \chi(D) \geq k+1$, moreover C is a hamiltonian circuit of D'.

Let u be a vertex of D', if $v_1, ..., v_{k-1}$ are in-neighbors of u in D', listed in such a way that $v_1, ..., v_{k-1}, u$ appear in the same order along C, the sub-circuit of $C + v_{k-2}u$ not containing v_{k-1} would have length at least k since it contains $v_1, ..., v_{k-2}$ and u in addition to the out-neighbor of u in C. This contradicts the minimality of C, so we conclude that every vertex has at most $k-2$ in-neighbors in D' and similarly at most $k-2$ out-neighbors in D'.

A *handle decomposition* of D' is a sequence $H_1, ..., H_r$ such that:

1. H_1 is a circuit of D'.
2. For $2 \leq i \leq r$, H_i is a *handle*, that is, a directed path in D' (with possibly the same end-vertices i.e. a circuit) meeting $V(H_1 \cup ... \cup H_{i-1})$ exactly at his end-vertices.
3. $D' = H_1 \cup ... \cup H_r$.

An H_i which is an arc is a *trivial handle*. It is well-known that r is invariant for all handle decompositions of D' (indeed, r is the number of arcs minus the number of vertices plus one, it is proved by a simple induction on r). However the number of nontrivial handles is not invariant. Let us then consider $H_1, ..., H_r$, a handle decomposition of D' with minimum number of trivial handles. Since the trivial handles does not add any new vertices, we can enumerate first the nontrivial handles, and so we can assume that $H_1, ..., H_p$ are not trivial and that $H_{p+1}, ..., H_r$ are arcs.

Let $D^o := H_1 \cup ... \cup H_p$. Clearly D^o is a strongly connected spanning subdigraph of D. Observe that since $\chi(D') > 3$, D' is not an induced circuit which means that $r > 1$, so $p > 1$

because otherwise a trivial handle would be a chord of H_1 so by shortcutting H_1 through this chord we get two non trivial non handles which contradicts the maximality of p.

We denote by $x_1, ..., x_q$ the handle H_p minus its end-vertices.

If $q = 1$, the digraph $D^o - x_1$ is strongly connected, and therefore $D' - x_1$ is also strongly connected. Moreover since $\chi(D') \geq k+1$ we have $\chi(D' - x_1) \geq k$. Thus by Bondy's theorem, there exists a circuit of length at least k in $D' - x_1$ that is shorter than C, a contradiction with the minimality of C.

If $q = 2$, x_2 is the unique out-neighbor of x_1 in D' because otherwise we would make two non trivial handles out of H_p, contradicting the maximality of p. Similarly, x_1 is the unique in-neighbor of x_2. Since the out-degree and the in-degree of every vertex is at most $k-2$, both x_1 and x_2 have degree at most $k-1$ in the underlying graph of D. Since $\chi(D) > k$, it follows that $\chi(D - \{x_1, x_2\}) > k$ because otherwise we can extend a good k-coloring of $D - \{x_1, x_2\}$ by giving each of x_1 (we can always find such a color since x_1 is adjacent to at most $k-1$ vertices) and then we do the same with x_2. Since $D - \{x1, x2\}$ is strongly connected, it contains, by Bondy's theorem, a circuit with length at least k, contradicting the minimality of C.

Hence, we may assume that $q > 2$. $\forall i \in \{1, ..., q-1\}$, by the maximality of p, the unique arc in D' leaving $\{x_1, ..., x_i\}$ is $x_i x_{i+1}$ (otherwise we would make two nontrivial handles out of H_p). Similarly, $\forall i \in \{2, ..., q\}$, the unique arc in D' entering $\{x_j, ..., x_q\}$ is $x_{j-1}x_j$. In particular, as for $q = 2$, x_1 has out-degree 1 in D' and x_q has in-degree 1 in D'.

The next paragraph is, word by word (with exception for the terminology), the last part of the proof in [2] **which contains an error**:

"Another consequence is that the underlying graph of $D' - \{x_1, x_q\}$ has two connected components $D_1 = D' - \{x_1, x_2, ..., x_q\}$ and $D_2 = D'[\{x_2, ..., x_{q-1}\}]$. Since the degrees of x_1 and x_q in the underlying graph of D' are at most $k-1$ and D' is at least $(k+1)$-chromatic, it follows that $\chi(D_1)$ or $\chi(D_2)$ is at least $k+1$. Each vertex has in-degree at most $k-2$ in D' and $d^+_{D_2}(x_i) \leq 1$ for $2 \leq i \leq q-1$, so $\Delta(D_2) \leq k-1$ and $\chi(D_2) \leq k$. Hence D_1 is at least $(k+1)$-chromatic and strongly connected. Thus by Bondy's theorem, D_1 contains a circuit of length at least k but shorter than C. This is a contradiction." [2]

The error is that there is no reason to say that $d^+_{D_2}(x_i) \leq 1$ for $2 \leq i \leq q-1$, in fact $x_i \nrightarrow x_j$ for $j > i$ but we can have $x_i \rightarrow x_j$ for $j < i$ and so we can have $d^+_{D_2}(x_i) > 1$, and therefore $\Delta(D_2)$ can be greater than $k-1$. So we will prove that $\chi(D_2) \leq k$ through another proof:

Let $D^i := D[\{x_i, ..., x_{q-1}\}]$ and let i be the minimum integer greater than 1 such that $\chi(D^i) \leq k$. Suppose that $i > 2$: since the unique arc in D entering $\{x_i, ..., x_q\}$ is $x_{i-1}x_i$ then we have $d^+_{D^{i-1}}(x_{i-1}) = 1$ and since $d^-_D(x_{i-1}) \leq k-2$ we have $d_{D^{i-1}}(x_{i-1}) \leq k-1$ and therefore $\chi(D^{i-1}) \leq k$ which contradicts the minimality of i. Then $i = 2$ and $\chi(D_2) = \chi(D^2) \leq k$. $\square$

The existence of good circuits directly implies the main theorem in the case of strongly

connected digraphs.

Corollary 3.5: *Let $k+l=n-1$ where $n \geq 4$ and let D be a strongly connected n-chromatic digraph then D contains a $P(k,l)$.*

Proof: Since $P(k,l)$ and $P(l,k)$ represent the same digraph and since $k+l=n-1 \geq 3$, we may assume that $l \geq (n-1)/2 \geq 3/2$ which means that $l \geq 2$. By *lemma 3.4* D contains an $(l+1)$-good circuit C, the chromatic number of the (strongly connected) contracted oriented multi-graph D/C is at least k, since otherwise we may use a good k-coloring of D/C to construct a good $n-1$-coloring of D: keep the colors of the vertices of $D-C$, and for the vertices of C we give one vertex the color of v_C and then we color the other vertices by l new colors. We conclude that $\chi(D/C) \geq k+1$ and by Bondy's theorem, D/C has a circuit of length at least $k+1$, and in particular the vertex v_C is the end of a path P of length k in D/C. Finally $P \cup C$ contains a $P(k,l)$. □

3.4 General case, first method (Addario-Berry et al)

Theorem 3.6: *Let $k+l=n-1 \geq 3$ and let D be an n-chromatic digraph. Then D contains a $P(k,l)$.*

Proof. We can assume that $l \geq k$, and therefore $l \geq 2$. Suppose to the contrary that D does not contain $P(k,l)$. Let F be a maximal spanning out-forest of D.
Consider the following coloring (Which we call *canonical coloring*) of D: for $1 \leq i \leq k-1$, the vertices of F_i are colored i, and for $i \geq k$, the vertices of F_i are colored j, where $j \in \{k,...,k+l\}$ and $j \equiv i \bmod l+1$. Since we colored D with less than n colors, this coloring can not be good. In particular, there exists an arc $v \rightarrow w$ from F_i to F_j where $i,j \geq k$ and $j \equiv i \bmod l+1$. By *Lemma 3.1 (1)*, we get a contradiction if $i<j$. Thus $j<i$, and by *Lemma 3.1 (2)*, we necessarily have $j=k$ and $i \geq k+l+1$. Since F is a maximal spanning out-forest we can find in F a directed path from w to v. In particular $F+vw$ has a circuit C of length at least $l+1$. If $\chi(D[C]) \leq l+1$ then C is $(l+1)$-good, if not, then by *Lemma 3.4*, it contains an $(l+1)$-good circuit. So in all cases we can find an $(l+1)$-good circuit which is disjoint from $F_1 \cup ... \cup F_{k-1}$.

We inductively define couples (D^i, F^i) as follows: Set $D^0 := D$, $F^0 := F$. Then, if there exists an $(l+1)$-good circuit C^i of $D^i - F_1^i \cup ... \cup F_{k-1}^i$, define $D^{i+1} := D^i - V(C^i)$ and let F^{i+1} be any maximal closure in D^{i+1} of $F_i - V(C_i)$.

With the previous definitions, we have $D^1 = D - V(C^0)$. This inductive definition certainly stops on some (D^p, F^p) where the canonical coloring of D^p is a good coloring.

At each inductive step, the circuit C^i must contain a vertex v^i of F_k^i, otherwise the union of C^i (which has length at least $l+1$) and a path of F^i starting at F_1^i and ending at C^i (which would have length at least k if C^i does not meet F_k^i) would certainly contain a $P(k,l)$. Let u^i the unique in-neighbor of v^i in F_{k-1}^i. $\forall j>i$, $l_{F^j}(u^i)=k-1$, since $l_{F^j}(u^i) \geq k-1$ because we apply successive elementary improvements, and $l_{F^j}(u^i)$ cannot be greater than $k-1$, otherwise u^i would be the end of a path P of length $k-1$ in $D-C^i$ and thus $Ci \cup P \cup u^i v^i$

would contain a $P(k,l)$. Thus every circuit C^i, $i=0,...,p-1$, has an in-neighbor u^i in F^p_{k-1}.

Observe that we cannot have any arc between two circuits C^i since they are disjoint and the length of each one is at least $l+1$, and if there is such an arc we get a $P(k,l)$ since $l \geq k$. Observe also that no vertex of C^i has a neighbor, (in- or out-), in any level F^p_j for any $j > k$ because otherwise we get a $P(k,l)$. Moreover, no vertex of C^i has an in-neighbor in F^p_k.

Let us call *bad vertices* the out-neighbors of the vertices of all C^i in F^p_k and *good vertices* the non-bad vertices in F^p_k. A bad vertex b cannot have in-neighbors in more than one circuit C^i, since the length of those circuits is at least $l+1$ and so joining two circuits C^i with b through two arcs towards it make a $P(k,l)$. Moreover b has at most l in-neighbors in C^i: Suppose to the contrary that $w_1,...,w_{l+1}$ are in-neighbors of b in C^i, enumerated with respect to the cyclic order of C^i such that w_1 is the first vertex w_j along C^i which appears after v^i (i.e. $C^i[v^i,w_1] \cap \{w_1,...,w_{l+1}\} = \{w_1\}$). Let P be the path of F^p starting at F^p_1 and ending at u^i. Now $P \cup u^iv^i \cup C[v^i,w_1] \cup w_1b \cup C[w_2,w_{l+1}] \cup w_{l+1}b$ contains a $P(k,l)$, a contradiction.

Let b is a bad vertex, we denote by S_b the set of *descendants* of b in F_p, i.e. the set of vertices x such that there is a path from b to x in F_p, including b itself.

We claim that every arc $x \to y$ entering S_b (i.e. $y \in S_b$ and $x \notin Sb$) in $D' := D - F^p_1 \cup ... \cup F^p_{k-1}$ is such that $y=b$ and $x \in C^i$. Indeed, suppose that $y \neq b$, y would be a strict descendant of b in F^p and then $l_{F^p}(y) > k$ and so $x \notin C^j$ $\forall j \in \{1,2,...,p-1\}$, thus $x \in F^p$. Let P_1 be the path in F^p (of length at least $k-1$) ending at x, let P_2 be the path in F^p starting at b and ending at y and let v be an in-neighbor of b in C^i. $P_1 \cup xy \cup C^i \cup vb \cup P_2$ would contain a $P(k,l)$, which gives a contradiction; so we conclude that $y=b$, if $x \in F^p$ we would have $x \to b$, $l_{F^p}(x) \geq l_{F^p}(b) = k$ without having any directed path from b to x which contradicts the fact that F^p is a spanning maximal out-forest. So we must have $x \notin F^p$ which means that there exist $0 \leq j < p$ such that $x \in C^j$. We must have $j=i$ since b cannot have in-neighbors in more than one circuit C^j.

We claim also that we have no arcs leaving S_b. Indeed, let $x \to y$ be an arc of D' such that $x \in S_b$ and $y \notin S_b$. If $y \in F^p$, there exists a path P_1 (of length at least $k4$) in F^p ending at y which does not meet S_b nor C^i. LetP_2 be the path in F^p which starts at b and ends at x, and let v be an in-neighbor of b in C^i. We can then find a copy of $P(k,l)$ in $P_1 \cup C^i \cup vb \cup P_2 \cup xy$. Thus $y \notin F^p$ and therefore it belongs to some C^j , but this is impossible since $l_{F^p}(y) \geq k$.

We resume:

- There is no arcs between different C^i's.
- Each C^i is adjacent to a unique vertex in D' which is bad.
- If b is a bad vertex, then the only arcs between S_b and $D' - S_b$ are those between b and a unique circuit C^i, we have at most l such arcs.

Let us color D with $n-1$ colors. Let D_1 be the subdigraph D induced by the vertices of F^p which are not in S_b for any bad vertex b. The canonical coloring of D_1 is good since all the vertices in D_1 of level k are good. We will extend this coloring for the other vertices of D

(which are vertices of some C^i, or descendants of some bad vertex).

Every C^i is $(l+1)$-good and thus $(l+1)$-colorable. Moreover, we have no arcs between any two circuits C^i, so we may color their vertices by the colors $k, k+1, ..., k+l$. This extension of the coloring is also good since the vertices whose level is at most $k-1$ are colored with colors $1, ..., k-1$, and the vertices of D_1 whose level is at least k are descendants of good vertices.

So it remains to extend the coloring for the descendants of bad vertices. Let b be a bad vertex, then b is adjacent (in D') to at most l vertices in some unique C^i, so we can properly choose a color c for b from the $l+1$ colors $k, k+1, ..., k+l$. Since the strict descendants of b are not adjacent to any vertex outside S_b, we properly color any descendant v of b with a color $c(v)$ in $\{k, k+1, ..., k+l\}$ such that $c(v) \equiv c + l_{f^P}(v) \mod l+1$. We get a good $n-1$-coloring of D, which is a contradiction. $\square$

3.5 General case, second method (El-Sahili and Kouider)

To prove *theorem 3.6*, we will use the following weaker result, proved by El-Sahili and Bondy:

Theorem 3.7 [15]: *For $n \geq 4$, every n-chromatic digraph contains a path $P(n-2, 1)$.*

We explain now the new method of El-Sahili and Kouider to prove *theorem 3.6*:

New proof of theorem 3.6. Let D be an n-chromatic digraph. Due to *theorem 3.7*, it is sufficient to prove that D contains any path $P(k,l)$ with $2 \leq k \leq l$ and $k+l = n-1$. Consider a maximal spanning out-forest F of D minimizing $u_k(F) = \sum_{j=1}^{k-1} |L_j(F)|$. The vertices in $U_i = L_i(F)$ are taken the color i for $1 \leq i \leq k-1$. For $i \leq l$, set $U_{k+i} = \cup_{r \geq 0} L_{k+i+r(l+1)}(F)$.

Step 1: Suppose to the contrary that D contains no path $P(k,l)$. Then U_i is a stable set for $i \neq k$. Indeed, this fact is trivial for $i \leq k-1$. If U_i is not stable for $i > k$, then there is an edge $uv \in G(D[U_i])$. Since vertices having the same level are not adjacent, we must have $l_F(u) \neq l_F(v)$, then $|l_F(u) - l_F(v)| \geq l+1$ and $min(l_F(u), l_F(v)) \geq k+1$, so by *lemma 3.1* D contains a path $P(k,l)$ which is a contradiction. if U_k is stable, we get $n-1$ stables which contradicts the fact that $\chi(D) = n$, then U_k is not stable. By *lemma 3.1* the only possible arcs in U_k are those with heads in $L_k(F)$. These vertices of $L_k(F)$ are said to be bad. It is clear that if v is a bad vertex then $T_v(F)$ contains a circuit of length at least $l+1$, and so each vertex in $T_v(F)$ is the end of a directed path of length l, and this means that:

There is no edge uw in $G(D)$ with $u \in T_v(F)$ and $w \notin T_v(F)$ such that $l_F(w) \geq k$ (1).

We get a contradiction if we give the uncolored vertices colors in $1, ..., k, ..., k+l$ to obtain a good $(n-1)$-coloring of D. By remark (1) This can be done separately on each $T_v(F)$ where v is bad. Let v be a bad vertex of F and suppose that F is chosen as above with a minimal number of bad vertices.

Step 2: Let $x, y \in N^-(v) \cap U_k$, we have $l_F(x) = l_F(y)$ since otherwise we will have $l_F(x) - l_F(y) \geq l+1$, and so $P_{vy} \cup P_{y'x} \cup xv \cup yv$, where $y' \in P_x$ and $l_F(y') = l_F(y) + 1$, contains a path $P(k,l)$. Set $h(v) = l_F(x) = l_F(y)$. A vertex $u \in D$ is said to be rich in F if $l_F(u) \geq k$

and $N(u) \cap L_i(F) \neq \phi$ for all $i \leq k-1$. If $N^-(v) \cap U_k$ contains no rich vertices, then each vertex $u \in N^-(v) \cap U_k$ can take a color $i \leq k-1$ such that $N(u) \cap L_i(F) = \phi$. A remainder vertex x takes the color $k \leq i \leq k+l$ if $x \in U_i$. We obtain an good $(n-1)$-coloring which is a contradiction. Similarly we verify that v is rich. Let u be a rich vertex in $N^-(v) \cap U_k$. we have $N^-(v) \cap U_k = u$. In fact if there is another vertex $w \in N^-(v) \cap U_k$, let s be the smallest integer such that $N^+(u) \cap L_s(F) \neq \phi$, we have $s \leq k$. Let $x \in N^+(u) \cap L_s(F)$. Since F is a maximal spanning out-forest then $x \in P_u$ which contains P_v. If $s = 1$, $ux \cup P_v \cup P_{vw} \cup wv$ contains a path $P(k,l)$. If $s > 1$, then let $y \in N^-(v) \cap L_{s-1}(F)$, y exists due to the minimality of s, $P_y \cup yu \cup ux \cup P_{xv} \cup P_{vw} \cup wv$ contains a path $P(k,l)$. The same argument proves that:

u is the unique rich in-neighbor of v with level greater than $n-1$ (2).

Denote by $\overline{v}$ the vertex u and by C_v the circuit $P_{vu} \cup uv$ and set $C_v = v_k v_{k+1} ... v_p v_k$ where $v_k = v$ and $v_p = u$. Note that there exist an integer f such that $l(C_v) = 1 + f(l+1)$. We show that v_{k+1} is a rich vertex: $N(v) \cap U_{k+1}$ must contain a rich vertex x, because otherwise we may give all the vertices in $N(v) \cap U_{k+1}$ an appropriate color in $\{1, 2, ..., k-1\}$ and then give v the color $k+1$, and the color i for remaining vertices in $T_v \cap L_i$. We get then a good $n-1$-coloring, a contradiction. Then we must have $x \in N^+(v) \cap L_{k+1}(F)$ by remark (2). If v_{k+1} is not rich then $x \notin C_v$. We show as above that $N(x) \cap L_i(F) = N^-(x) \cap L_i(F)$ for all $i \leq k-1$: If $\exists s, N^+(x) \cap L_i(F) \neq \phi$, we may suppose that s is minimal, let $y \in N^+(x) \cap L_s(F)$. If $s = 1$, $xy \cup P_v \cup C_v$ contains a path $P(k,l)$. If $s > 1$, then let $y' \in N^-(x) \cap L_{s-1}(F)$, y' exists due to the minimality of s, $P_{y'} \cup y'x \cup xy \cup P_{yv} \cup C_v$ contains a path $P(k,l)$.

If $zw \in E(G(D))$ with $w \in T_v - T_x$ and $z \in T_x$, we have $w = v$ and $z = x$: Suppose to the contrary that $z \neq x$, since $V(C_v) \subset V(T_v - T_x)$ w is the end of a directed path Q_w of length l in $T_v - T_x$. Let $y \in N^-(x) \cap L_{k-1}(F)$ so $P_y \cup yx \cup P_{xz} \cup Qw \cup wz$ contains a path $P(k,l)$ regardless of the orientation of wz. So we $z = x$, but this means that $w \to z$ because otherwise $P_y \cup yx \cup xw \cup Qw$ would again contain a $P(k,l)$. $w \to z$ means that either $l_F(w) < l_F(z)$ or $z \in T_w$, the latter case does not hold since $w \in T_v - T_x$ and $z \in T_x$, so $l_F(w) < l_F(z)$ and thus w is necessarily v. We conclude that vx is the only edge between $T_v - T_x$ and T_x. (3)

Color a vertex $z \in T_x \cap U_i$ by the color $i+1$ if $i < n-1$ and by the color k if $i = n-1$. We do the same with any other rich neighbor of v in U_{k+1}. We give the other vertices of $N(v) \cap U_{k+1}$ appropriate colors from $\{1, ..., k-1\}$, v is then colored by $k+1$ and each remainder vertex $z \in U_i$ $(k+1 \leq i \leq k+l)$ is colored by the color i. We get an good $(n-1)$-coloration of D, which is a contradiction.

So v_{k+1} is a rich vertex verifying $N(v_{k+1}) \cap L_i(F) = N^-(v_{k+1}) \cap L_i(F)$ for all $i \leq k-1$. Let $F_1 = F + yv_{k+1} + uv - vv_{k+1} - xv$ where x is the predecessor of v in F and $y \in N^-(v_{k+1}) \cap U_{k-1}$ and let F' be a maximal closure of F_1. Since $uk(F)$ is minimal, then $l_{F'}(z) = l_F(z)$ if $l_F(z) \leq k-1$. This proves that $L_k(F') = (L_k(F) \setminus \{v_k\}) \cup vk+1$ and v is still rich in F' with $l_{F'}(v) \geq p \geq n$, but v is an in-neighbor of v_{k+1}, then $\overline{v_{k+1}} = v$ and $h(v_{k+1}) \geq h(v)$. By supposing that F is chosen such that $\sum_{w \text{ is bad}} h(w)$ is maximal, we get $h(v_{k+1}) = h(v)$. This gives $l_{F'}(v) = l_{F_1}(v) = p$. Another important fact can be easily verified is that $l_{F'}(vs+1) = s$ for $k \leq s \leq p-1$. Hence $C_{vk+1} = C_v$. We repeat the same reasoning as above to prove that $v_{k'}$ $(k \leq k' \leq p)$ is also a rich vertex verifying $N(v_{k'}) \cap L_i(F) = N^-(v_{k'}) \cap U_i$ for all $i \leq k-1$.

This can be verified by a simple induction for all the vertices in C_v.

Step 3: If $\chi(D[C_v]) \geq l+2$, then by *theorem 3.7* $D[C_v]$ contains a path $P(l,1)$. This path can be completed to obtain a path $P(k,l)$ by adding $T_{v''} \cup v''v'$, where v' is the end-vertex of the $P(l,1)$ corresponding to the block of length 1 which is rich and v'' is an in-neighbor of v' of level $k-1$. Then we conclude that $\chi(D(Cv)) \leq l+1$.Color C_v by the $l+1$ colors $\{k,...,k+l\}$.

If C_v contains exactly $l+2$ vertices (i.e. $f=1$), then at least two of the vertices of C_v are not adjacent, we may suppose without loss of generality that $vv_j \notin E(G(D))$ since any vertex of C_v can take the level k in some convenient maximal spanning out-forest of D. We give each vertex v_s, $s \neq k$, the color s. Let $x \neq v_j$ be a rich vertex in $N(v) \cap U_j$ then we must have $x \in N^+(v) \cap L_j(F)$, otherwise we would use x (as above) to make a directed path of length k ending at v, and intersecting C_v only at v, so by adding an appropriate directed path of C_v we get a $P(k,l)$. We prove as above (as in (3)) that if $zw \in E(G(D))$ with $w \in T_v - T_x$ and $z \in T_x$, we have $z = x$, $w \rightarrow z$ and $l_F(w) < j$.Color a vertex $z \in T_x \cap U_i$ by the color $i+1$ if $i < n-1$ and by the color j if $i = n-1$. We do the same for all rich vertices in $N(v) \cap U_j$ and the other non-reach vertices in $N(v) \cap U_j$ are colored by appropriate colors from $\{1,...,k-1\}$. The vertex v is colored by j and each remainder vertex $z \in U_i$ is colored by the color i, $k+1 \leq i \leq k+l$. We get a good $(n-1)$ coloring, which is a contradiction.

We conclude that $l(C_v) > l+2$ (i.e. $f > 1$), so $p > n$ and $l(C_V) = 1 + f(l+1) \geq 1 + 2(l+1)$. If we consider two vertices v_s and v_t in C_v with $s < t \leq p$. Since $l(C_V) =\geq 1 + 2(l+1)$, then C_v may be viewed as the union of two directed paths $Q_{v_s v_t}$ and $Q_{v_t v_s}$, such that one of them, say P, is of length at least $l+1$. Set $S_{v_j} = T_{v_j} - T_{v_{j+1}}$ for $k \leq j \leq p-1$ and $S_{v_p} = T_{v_p}$. If $x \in S_{v_t}$ and $y \in S_{v_s}$ are such that $xy \in E(G(D))$ and $\{x,y\} \neq \{v_s, v_t\}$. If $s \neq k$ or $y \neq v$, $P \cup P_{v_t x} \cup xy \cup P_w \cup wv_s \cup P_{v_s y}$ would contain a path $P(k,l)$ regardless of the orientation of xy, where w is the in-neighbor of v_s in L_{k-1}. So we must have $s = k$ and $y = v$. If $t \neq p$, $P \cup P_{v_t x} \cup xy \cup P_z \cup zu \cup uv \cup P_{vy}$ would contain a $P(k,l)$ regardless of the orientation of xy, where z is the in-neighbor of $u = v_p$ in L_{k-1}. So we must have $t = p$ and $y = v$.

Color C_v by the $l+1$ colors $\{k,...,k+l\}$ such that v is colored k and u is colored $k+1$. For all $w \in C_v$ of color $j = k+r$ we color each vertex $x \in L_m(S_w)$ by the color $k+h$ with $h \leq l$ and $h \equiv m+r-1 \mod (l+1)$. We claim that the vertices in S_u of color k cannot be adjacent to v: If $w \in S_u$ is of color k then $l_F(w) \geq p+l$ and so if w is adjacent to v, then $P_v \cup vw \cup P_{vw}$ contain a $P(k,l)$ if $v \rightarrow w$ and $P_{vu} \cup uv \cup P_{uw} \cup wv$ contain a $P(k,l)$ if $w \rightarrow v$. Then this coloring is a good $n-1$-coloring of D, which contradicts the fact that $\chi(D) = n$.
□

3.6 Conclusion

We have presented in this chapter the problem of finding paths with two blocks in n-chromatic digraphs. We have proven with two methods that for $n \geq 4$, we can find any oriented path of length $n-1$ with two blocks in any n-chromatic digraph. What if we have more than two

blocks?

We conclude this chapter by stating this new conjecture of El-Sahili:

Conjecture 3.8 [3]: *For $n \geq 8$, every n-chromatic digraph contains any oriented path of length $n-1$.*

In fact this conjecture generalizes Rosenfeld's conjecture which states that every tournament of order n contains any oriented path of order $n-1$, which was proved by Havet and Thomassé with three exceptions which are tournaments of order 3,5 and 7. The condition $n \geq 8$ is therefore necessary due to these three exceptions.

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